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THE THEORY OF IMAGES IN THE REPRESENTATION OF FUNCTIONS.*

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1. If u be a function of z, and if z describe any path, u will describe another path which is called the *image* of the former with reference to the given function.

The problem constantly before us in the following discussion, is that of ascertaining points in the u-plane which correspond to given points in the z-plane (thereby *representing* the function), without involving needless computation.

2. The *transforming equation* is the relation subsisting between u and z. It will usually be indicated thus:

$$\Phi(u, z) = 0, \quad \text{or} \quad u = \varphi(z).$$

Either Φ or φ may be called the *transforming function*.

3. The *path*, when not otherwise specified, is the course of the independent variable. Its equation will usually be written

$$f(x, y) = 0.$$

We shall indicate the equation of the image, with respect to Φ, thus:

$$\mathbf{I}^{f(x,y)}_{\Phi(u,z)} = 0, \quad \text{or} \quad \mathbf{I}^{f}_{\Phi} = 0.$$

4. If u be a many-valued function of z, m denoting the order of multiplicity; and if z be a many-valued function of u, n denoting the order of multiplicity; then to each value of z there correspond, in general, m distinct values of u, and to each value of u correspond n of z.

In this case either the path, or the image, or both, *may* be *multiform*.

5. By supposing f to contain a variable parameter, we obtain a *system of paths*. The corresponding courses of u constitute an *image of the system*.

When for a particular value of the parameter, the degree of the equation of the image is lowered in respect to either variable or to both variables (the curve

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then breaking up into curves of lower degree), this line will be called a *path of reduction*, and the corresponding course of u, a *reduced image*.

EXAMPLES OF PATHS OF REDUCTION (Φ REAL).

(1.) The axis of x for the system $y=\beta$. For, if we equate the imaginary part of $\Phi(u, z)$ to zero, every term must contain a y or Y. Hence when $y=0$, Y will be a factor of the equation just formed.

(2.) If the order of each of the terms of Φ be odd (and the constant term wanting) then the axis of y is a line of reduction in the system $x=\alpha$.

(3.) If the variables u, z be separate, if u enter by odd powers only, and if the system be the real part of the function of z equated to k, that value of k which cancels the constant in Φ will give a curve of reduction.

(4.) On the other hand, if u and z be separate and the purely imaginary part of the function of z equated to k', $k'=0$ will always give a curve of reduction.

(5.) Every quardratic function with real coefficients will be shown to have a line of reduction parallel to the y-axis. Art. 21.

6. *Translation* is the result of substituting for u, $u'-\nu$, and for z, $z'-\nu'$, in the transforming equation, and simultaneously for x, $x'-\nu_1'$, and for y, $y'-\nu_2'$, in the equation of the path. ν, ν' are any constants; $\nu_1'+i\nu_2'=\nu'$. That is, if we move the origin $O_{z'}$ to the point whose affix is ν', any path of z' becomes a path of z referred to O_z. If we move $O_{u'}$ to the point ν, then any course of u' becomes a course of u referred to O_u.

7. *Imaginaries* is a generic term applied to all quantities not real; these latter may be either *complex quantities*, or *pure imaginaries*.

8. For the sake of brevity, we shall denote the real part of any expression or function F, by $\mathfrak{r}F$, and the purely imaginary part by $i\mathfrak{p}F$.

9. Expressions whose terms are homogeneous in each of two sets of letters, as $X^m y^n$, $X^{m-r} Y^r x^{n-s} y^s$, etc., may be denoted thus:

$$(X, Y)^m \underline{\ } (x, y)^n.$$

10. *If z describe any infinitesimal element, u describes a similar one.*

Let the arguments of dz and du be ω and χ, respectively; σ, ρ their moduli.*

Then $$dz=\sigma e^{i\omega}, \quad du=\rho \varepsilon^{i\chi};$$

$$\therefore \frac{du}{dz}=\frac{\rho}{\sigma} e^{i(\chi-\omega)}. \tag{1}$$

But $$\frac{du}{dz}=\frac{\partial X}{\partial x}+i\frac{\partial Y}{\partial x}=\frac{\partial Y}{\partial y}-i\frac{\partial X}{\partial y}. \tag{2}$$

*Hoüel, *Calcul Infinitésimal*, Vol. 3. p. 249.

From (1) and (2) we get

$$\frac{\rho}{\sigma} = \sqrt{\left[\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2 \right]}, \quad \chi - \omega = \tan^{-1} \frac{\frac{\partial Y}{\partial x}}{\frac{\partial X}{\partial x}}.$$

Hence $\chi - \omega$ depends only upon the position of z, and is always the same for that point regardless of the direction in which dz is taken. If, then, we increase or diminish the argument of dz by any given amount, the argument of du will be increased or diminished by an equal amount. Hence all angles of the infinitesimal elements, described by z, are preserved in the images of these paths.

If $u = \varphi(z)$, and if $v = \psi(u, z)$, then of course, the elements described by v are similar to those described by u or z.

11. By taking the inverse function along the image, we get the original curve or path as an image of the image.

The truth of this proposition is obvious. Since any point z, operated upon by φ gives u, or $\varphi(z)$, if we operate upon this result by φ^{-1}, the two operators mutually destroy each other.

12. If we develop $u = \varphi(z)$ by Taylor's theorem, we shall have

$$u = \varphi(x) + \frac{iy}{1}\frac{\partial\varphi(x)}{\partial x} + \frac{(iy)^2}{2!}\frac{\partial^2\varphi(x)}{\partial x^2} + \ldots + \frac{(iy)^n}{n!}\frac{\partial^n\varphi(x)}{\partial x^n};$$

$$\therefore X = \varphi(x) - \frac{y^2}{2!}\frac{\partial^2\varphi(x)}{\partial x^2} + \frac{y^4}{4!}\frac{\partial^4\varphi(x)}{\partial x^4} - \ldots \equiv \cos\left(y\frac{\partial}{\partial x}\right) . \varphi(x), \qquad (1)$$

$$Y = y\frac{\partial\varphi(x)}{\partial x} - \frac{y^3}{3!}\frac{\partial^3\varphi(x)}{\partial x^3} + \frac{y^5}{5!}\frac{\partial^5\varphi(x)}{\partial x^5} - \ldots \equiv \sin\left(y\frac{\partial}{\partial x}\right) . \varphi(x). \qquad (2)$$

If we eliminate x and y from (1), (2), and $f(x, y) = 0$, the resultant equated to zero will be the equation of the image.

SYMMETRY; φ REAL.

13. *If $f(x,y) = 0$ be symmetric about the x-axis, then is the image symmetric about the X-axis.*

This is equivalent to the statement that if

$$\varphi(x + iy) = X + iY,$$

then

$$\varphi(x - iy) = X - iY.$$

When the path is symmetric about the y-axis, then if φ be an $\frac{\text{odd}}{\text{even}}$ function, I is symmetric about the $\frac{\text{Y-axis}}{\text{X-axis}}$.

THE ORDER OF I, WHEN f AND Φ ARE RATIONAL ALGEBRAIC FUNCTIONS.

14. *Its limit.* Let m denote the highest power in which u enters Φ, n the highest power of z, and let p be the order of f.

Then the (x,y)-resultant of

$$\mathfrak{r}\Phi = 0, \quad \mathfrak{p}\Phi = 0, \quad f = 0,$$

i. e., the resultant got by eliminating (x,y), is of the pnth degree in the coefficients of each of the first two equations. But these coefficients contain X, Y to an order not greater than m; hence the order of I is not greater than $2mnp$.

15. *General method of ascertaining the order.*

Let
$$U_\alpha z^n + U_\beta z^{n-1}v + \ldots + U_\tau v^n = 0, \tag{1}$$

represent the general transforming equation; let $f(x, y, v) = 0$ be the equation of the path. v is introduced for the sake of homogeneity and is always understood to be equal to unity. U_α, U_β, etc., are polynomials of the orders α, β, etc. in u. Expanding (1), and equating $\mathfrak{r}\Phi$ and $\mathfrak{p}\Phi$ separately to zero, we have

$$(\grave{}X, Y)^\alpha \underline{} (x,y)^n + (\grave{}X, Y)^\beta \underline{} (x,y)^{n-1}v + \ldots + (\grave{}X, Y)^\tau v^n = 0,$$

$$(\grave{}X, Y)'^\alpha \underline{} (x,y)'^n + (\grave{}X, Y)'^\beta \underline{} (x,y)'^{n-1}v + \ldots + (\grave{}X, Y)'^\tau v^n = 0.$$

The grave accent is used to indicate that the function is not in general homogeneous in X, Y.

Suppose that the coefficients of all terms of $\mathfrak{r}\Phi$, $\mathfrak{p}\Phi$ containing v to a given power be affected with a suffix equal to this power, then the (x,y)-resultant will be a homogeneous function in these coefficients of order $2np$ and weight n^2p. Let a_0 denote any coefficient of v^0 containing X, Y to the α power; a_1, any coefficient of v containing X, Y to the β power, etc. Form out of the a's all possible terms having the required order and weight. Observe the order of each a (regarded as a function of X, Y) that enters any given term as a factor. Let θ denote this order, and let t be the number of times that it enters as a factor; then

$$\Sigma\,(t\theta)$$

is the order of the particular term in question. The highest order found, when the successive terms are thus treated, is the order of I.

16. *To find the degree to which a_r may enter the resultant.*

Let us first consider the special case where $p = 1$, and let t denote the degree in question. Then, since no subscript can exceed n, we have (because weight $\div$ order $=$ subscript),

$$\frac{n^2 - rt}{2n - t} \overline{\overline{<}}\, n; \quad \therefore t \overline{\overline{<}} \frac{n^2}{n - r}.$$

If we use E to indicate that only the integral part of the quantity is taken, we have

$$t = E\left(\frac{n^2}{n-r}\right). \tag{1}$$

This formula holds true whenever $n \geqq 2r$. When $n < 2r$, we must use the obvious formula

$$t = E\left(\frac{n^2}{r}\right). \tag{2}$$

Reasoning as before, we have in general

$$t = E\left(\frac{n^2p}{n-r}\right), \quad n \geqq 2r, \tag{3}$$

$$t = E\left(\frac{n^2p}{r}\right), \quad n \leqq 2r. \tag{4}$$

If Φ be a homogeneous function in u and z of order n, then the image of any curve of the pth degree is of the order n^2p. For the subscripts of the a's in this case denote their orders in X, Y. Hence the weight of any term will be the order of I. But we know this to be n^2p. So, if $\alpha, \beta, \ldots \tau \leqq 0, 1, \ldots n$, respectively, even if Φ be not homogeneous, the degree of I is at most n^2p.

Φ OF THE FIRST DEGREE.

17. When Φ is of the first degree in u and z, the *nature* of any path transformed will be preserved. In other words, the path will at most be translated, rotated, and altered in scale.

Let the transforming equation be

$$u = kz + c.$$

Take any point z_1 upon the curve $f = 0$. If we multiply it by the affix of k, we increase its argument by a, the argument of k, and multiply its modulus by l, the modulus of k. We then translate the resulting point a distance c. So, for every point on $f = 0$.

Let $z = ku$, and suppose the modulus of k to be equal to unity. Then,

$$x = X\cos a - Y\sin a, \tag{1}$$

$$y = X\sin a + Y\cos a, \tag{2}$$

or

$$z = e^{ia}u. \tag{3}$$

Thus the well-known formulae (1) and (2) can be written in the form (3).

Φ OF THE SECOND DEGREE.

18. We shall begin the study of these functions by examining cases which

are special, either because certain coefficients are wanting, or because the coefficients are real, or for both reasons.

(1.) Suppose that $a, h = 0$ in the general equation

$$au^2 + bz^2 + c + 2fz + 2gu + 2huz = 0. \qquad (1)$$

By translation, this equation may now be written in the form

$$u' = b'z'^2. \qquad (2)$$

Consider the equation $$u'' = z'^2,$$

or $$R'' \text{ cis } \theta'' = \rho'^2 \text{ cis } 2\theta';$$

$$\therefore R'' = \rho'^2, \quad \theta'' = 2\theta'.$$

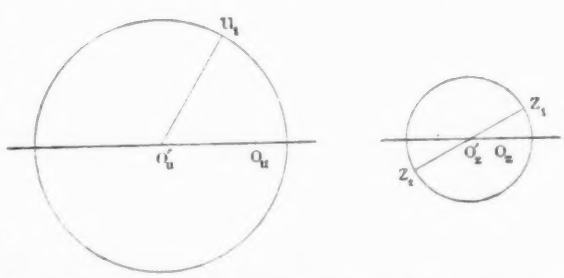
Now cause z' to describe a circle about O_z' with a radius ρ_1'; u'' will describe a circle about O_u' of radius $\rho_1'^2$. $\theta'' = 2\theta'$; hence, when u'' has made a complete revolution, z' has made but half of one. We have, then, for each value of u'' two of z' (which differ only in sign) in our representation. u' is obtained by multiplying the modulus of u'' by that of b', and increasing the argument of u'' by that of b'.

EXAMPLE. $$u + 2 = z^2 + 2z.$$

Here $\quad u + 3 = (z + 1)^2, \quad u'' = u' = u + 3, \quad b' = 1, \quad z' = z + 1.$

That is, if we measure the co-ordinates of z_1, z_2 referred to O_z, and of u_1 referred to O_u, the values thus obtained should satisfy the equation

$$u + 2 = z^2 + 2z.$$



In the case of real coefficients, the two systems of straight lines $x = \alpha, y = \beta$ are transformed into two systems of confocal parabolas, all axes of the curves of both systems lying along a line parallel to the axis of X.

(2.) Suppose $b, h = 0$; the equation becomes, by translation, of the form

$$u^2 = f'z.$$

Obviously, this may be represented by aid of circles, as was the equation of the form $u = b'z^2$ in the last article.

If z move along any straight line, u always describes an equilateral hyperbola.

For, let $$f' = f_1' + if_2',$$

then $$X^2 - Y^2 = f_1'x - f_2'y, \qquad (1)$$

$$2XY = f_2'x + f_1'y. \qquad (2)$$

Substituting for y in (1), (2) its value from the equation of the line, and eliminating x, we see that the axes of the hyperbola are equal.

In the case of real coefficients, the image of $x = a$ is

$$\frac{X^2}{f'a} - \frac{Y^2}{f'a} = 1,$$

and that of $y = \beta$ is

$$2XY = f'\beta.$$

(3.) Suppose $a, b = 0$; the general equation now becomes, by translation, of the form

$$uz = c'.$$

Suppose $c' = 1$; then $u = \frac{1}{z}$, or

$$R \text{ cis } \Theta = \frac{1}{\rho} \text{ cis } (-\theta);$$

$$\therefore R = \frac{1}{\rho}, \quad \Theta = -\theta.$$

Hence, *the substitution $u = \frac{1}{z}$ gives the inverse of any curve along which z is taken at the same time inverting it.*

Multiply this locus by c' and we have the required curve.

(4.) If $h = 0$, the general equation may be written in the form

$$a'u^2 + b'z^2 = c'.$$

For the systems $x = a$, $y = \beta$ the images are always of the form (if the coefficients are real)

$$Y = \pm k\sqrt{\left(\frac{X^2 \pm C^2}{X^2 \pm B^2}\right)},$$

where the $\pm$ signs under the radical depend upon the case considered.

This curve admits of a mechanical construction. We use for this purpose two right-angled triangles having a side in common whose length is X.

GENERAL CASE WITH REAL COEFFICIENTS.

19. Having by translation deprived the equation of terms of the first degree, we may write it in the form

$$au^2 + 2huz + bz^2 = c.$$

If, now, z be made to move along the line $x = a$, the image becomes of the form

$$Y^2 = \frac{F_4(X)}{F_2(X)},$$

wherein F_4 and F_2 denote polynomials of the fourth and second degrees, respectively.

For $y = \beta$, I has the same form after interchanging X and Y.

Hence, *the images of all lines whose equations are $x = \alpha$, or $y = \beta$ can be easily constructed when Φ is a real quadratic function.*

GENERAL CASE.

20. By linear substitution, we can put $\Phi_2(u, z) = 0$ into the form

$$u'z' = c.$$

Now, if z' be taken along any straight line, u' will describe a circle, as follows from 18 (3). Hence, this auxiliary image is easily constructed. Suppose such a system to have been thus constructed. Then u and z are obtained from it by the relations

$$u = \lambda u' + \mu z' + \nu,$$
$$z = \lambda' u' + \mu' z' + \nu'.$$

Since the courses of u are images of the courses of z with reference to the original function, we have a representation as required.

COROLLARY. z may now be taken along more convenient systems and, by referring to the u- and z-charts already constructed, the corresponding course of u be determined.

EXAMPLE. $$4u^2 + 6z^2 - 11uz + 14u - 13z + 256 = 0. \quad (1)$$

Here, $$u = 2u' + 3z' + 1, \quad (2)$$
$$z = u' + 4z' + 2; \quad (3)$$
$$\therefore u' = \tfrac{1}{5}(4u - 3z + 2), \quad z' = \tfrac{1}{5}(2z - u - 3); \quad (4)$$
$$u'z' = 10.$$

Let z' move over two systems of right lines whose equations are $x = \alpha, y = \beta$. In the diagram, α varies from $\pm \frac{1}{2}$ to ± 4 and β has the same limits. When the paths of z' lie in the first angle, since $\theta' = -\theta'$, their images lie in the fourth angle, etc.

From these two charts we construct Figs. 3 and 4, by aid of equations (2) (3), using the first angle of Fig. 1 and the fourth angle of Fig. 2. Then the curves in Fig. 4 are the images of the curves in Fig. 3 with reference to equation (1).

LINE OF REDUCTION.

21. *The general quadratic equation with real coefficients has a line of reduction parallel to the y-axis.*

This line passes through the two critical points, or midway between them.

From what has been said concerning *order of images*, it can be easily shown

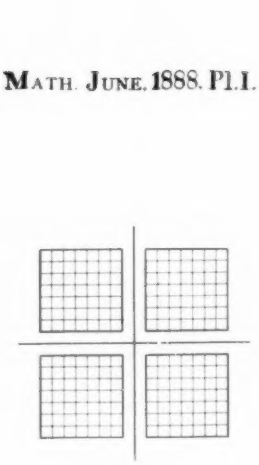


Fig. 1.

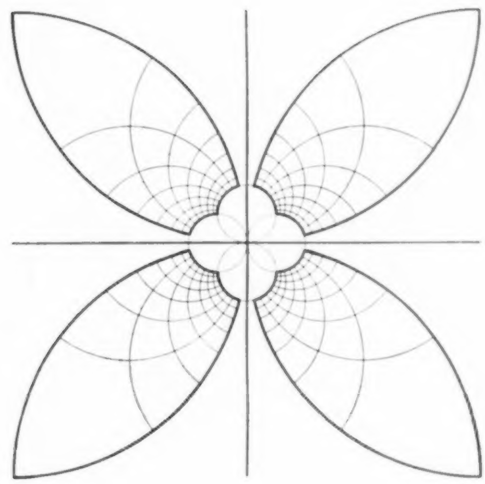


Fig. 2.

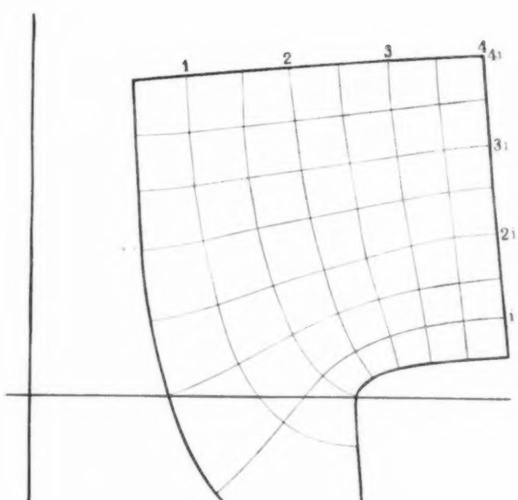


Fig. 3.

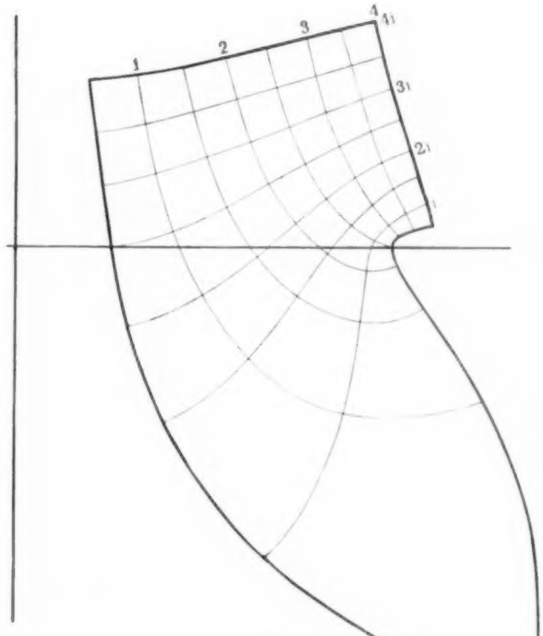


Fig. 4.

$I^{f_1}_{\Phi} = 0$ is, in general, of the fourth degree in X, Y. From $\mathfrak{p}\Phi = 0$, we have

$$y = -\frac{aXY + hYx + gY}{hX + bx + f} = -\frac{g}{h}Y$$

when $\frac{a}{h}(bx + f) = hx + g$; i. e., when

$$x = \frac{gh - fa}{ab - h^2}.$$

$$\mathfrak{r}\Phi = a(X^2 - Y^2) + b(x^2 - y^2) + 2h(Xx - Yy) + 2gX + 2fx + c = 0.$$

In this equation, substitute the above values for x and y. It will become the equation of

An hyperbola, if	$ab > h^2$;
An ellipse, if	$ab < h^2$;
A parabola, if	$ab = h^2$.

If we eliminate u between Φ and $\frac{\partial \Phi}{\partial u}$, we have

$$z = \frac{gh - af}{ab - h^2} \pm \frac{\sqrt{-a \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}}}{ab - h^2}.$$

The first part of the above expression is always real and represents the x-co-ordinate of a point midway between the two critical points. In case the second part is imaginary, this line passes through the points themselves. *Q. E. D.*

If the terms of the first degree be wanting, the line is the y-axis.

Φ OF THE FORM $\chi(u) - \psi(z)$.

22. Let us suppose χ of the mth degree and ψ of the nth. The order of I evidently depends upon the number of times that a_n can enter any term of the resultant as a factor. From (4) Art. 16, we see that it is np. But a_n is a function of the mth order in X, Y. Hence the order of I is mnp.

In the present case, the relation between u and z may be advantageously shown by a pair of systems representing the course of z and another pair representing that of u whose equations in either case shall be of the form

$$r_1 r_2 r_3 \ldots = c, \qquad = F(r_1, r_2, r_3, \ldots), \tag{1}$$

$$\theta_1 + \theta_2 + \theta_3 + \quad . \ . \ . = a + 2k\pi, = G(\theta_1, \theta_2, \theta_3, \ldots); \tag{2}$$

wherein $r_1, r_2, \ldots, \theta_1, \theta_2, \ldots$ are the moduli and the arguments of the m or n factors into which $\chi(u)$ or $\psi(z)$ is resolvable.

It is easy to draw the normal at any point of (1), and the tangent at any point of (2).

For we have only to lay off on $r_1, r_2 \ldots$ distances proportional to

$$\frac{\partial F}{\partial r_1}, \quad \frac{\partial F}{\partial r_2}, \quad \ldots ,$$

respectively, and, regarding them as forces, find the resultant. This gives the direction of the normal.

The direction of the tangent to (2) for a given point is found by taking distances on the r's proportional to

$$\frac{1}{r_1}\frac{\partial G}{\partial \theta_1}, \quad \frac{1}{r_2}\frac{\partial G}{\partial \theta_2}, \quad \ldots ,$$

and by finding the resultant as before.

Since
$$\frac{\partial F}{\partial r_1} = r_2 r_3 \ldots = \frac{c}{r_1},$$

and
$$\frac{1}{r_1}\frac{\partial G}{\partial \theta_1} = \frac{1}{r_1}, \text{ etc.},$$

the two systems cut each other orthogonally, as is otherwise evident. For (1), (2) may represent either $\chi(u)$ or $\psi(z)$ as an auxiliary variable $s, = \chi(u) = \psi(z)$, describes a system of concentric circles and a system of straight lines through their centre. The s-pair of systems being orthogonal, so is the u-pair, or the z-pair, and therefore (1) and (2).

All asymptotes to (2) pass through the centre of gravity of the zeros.

Let the point of contact of the tangent be at infinity; then $r_1, r_2 \ldots$ become as $1 : 1 : \ldots$. Hence, we may regard them as representing equal parallel forces acting upon the zeros. But the resultant of such forces passes through the centre of gravity of the zeros where these points are regarded as material and all equal in mass.

For a real function, this centroid will lie on the x-axis.

When the points are three in number, their centre of gravity coincides with that of a plane triangle defined by them; when m in number, the co-ordinates are

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots}{m},$$

$$\bar{y} = \frac{y_1 + y_2 + y_3 + \ldots}{m}.$$

Equation (1), of course, approaches the equation of a circle, where the m asymptotes to (2) are radii dividing it into m compartments.

23. Resume the equation

$$\chi(u) = \psi(z) = s,$$

where s is an auxiliary variable. From what has just been said, it is obvious that when s describes a circle whose radius is infinity, the argument of u is increased by $2\pi/m$ and that of z by $2\pi/n$, for each revolution of s. Since s describes $\frac{m}{n}$ circumferences while $\frac{u}{z}$ describes one (and this is true for any path enclosing *all* the zeros) to every value of s correspond $\frac{m}{n}$ distinct values of $\frac{u}{z}$; and these m or n values form in the limit a circular system at infinity.

Now let s describe a very small curve about the origin; $\frac{u}{z}$ will describe a similar curve about each of its $\frac{m}{n}$ zeros. If, however, the zeros be not simple, the above statement holds true for circular paths only.

Φ OF THE FORM $u^m - \psi(z)$ or $\chi(u) - z^n$.

24. In this case, either the path, or the image, becomes a circle, which simplifies the representation.

In the case where χ or ψ is a quadratic function, $\chi(u) - \psi(z)$ can always by translation be put into the form

$$u'^2 = \psi(z) + c,$$

or

$$c' + \chi(u) = z'^2.$$

25. If we wish to avoid vectorial co-ordinates, we may suppose s to describe the two systems $x' = k$, $y' = k'$, where $s = x' + iy'$. Then z will describe the pair of systems whose equations are

$$\mathfrak{r}\psi = k, \quad \mathfrak{p}\psi = k';$$

and u,

$$\mathfrak{r}\chi = k, \quad \mathfrak{p}\chi = k'.$$

If one of the functions, say ψ, be of the form z^n, then the two systems described by z have for their equations

$$\frac{1}{\rho^n} = \frac{1}{k}\cos n\theta, \tag{1}$$

$$\frac{1}{\rho^n} = \frac{1}{k'}\sin n\theta. \tag{2}$$

These curves are inverses of the well-known curves

$$r^n = a^n \cos n\theta,$$

$$r^n = a'^n \sin n\theta.$$

The asymptotes to (1) and (2) pass through the origin, dividing the region about it into $2n$ equal angular compartments. The angles which the asymptotes to (1) make with the x-axis are $\pi/2n$, $3\pi/2n$, $5\pi/2n$, . . . ; the same for (2), 0, $2\pi/2n$, $4\pi/2n$, $6\pi/2n$, ...

Equation $\binom{(1)}{(2)}$ will give n hyperbolic branches for $\pm\frac{k}{k'}$ and n conjugate ones for $\frac{-k}{-k'}$.

Φ OF THE THIRD DEGREE.

26. By means of the linear substitution,

$$u = \lambda u' + \mu z' + \nu,$$
$$z = \lambda' u' + \mu' z' + \nu',$$

we can always put $\Phi = 0$ into the form $\chi(u') = \psi(z')$. We have at our disposal four effective constants; therefore, after depriving Φ of the three terms containing u and z, we may impose one more condition. Let this be such a relation among the coefficients of χ or ψ that, by the addition of a constant to both sides of our equation, χ or ψ shall become a perfect cube. Then

$$(u' - \lambda)^3 = \psi(z') + k, \quad \text{or } u''^3 = \psi'(z').$$

Here, again, it will be advantageous for our pair of auxiliary systems to consist of 1° a system of concentric circles, and 2° their radii.

The equation $\chi(u') = \psi(z')$ will become

$$u'^3 = \psi'(z'),$$

by the above substitution, whenever the invariant S of a ternary cubic vanishes. For if

$$\begin{aligned}\Phi(u, z) \equiv\ & au^3 + bz^3 + c\nu^3 \\ & + 3a_2u^2z + 3a_3u^2\nu + 3b_1z^2u + 3b_3z^2\nu + 3c_1\nu^2u + 3c_2\nu^2z \\ & + 6muz\nu,\end{aligned}$$

then $$a_2, a_3, b_1, c_1, m = 0,$$

which values substituted in S make it zero.

If Φ be homogeneous in u and z, we may separate the variables by the substitution,

$$u = \lambda u' + \mu z',$$
$$z = \lambda' u' + \mu' z'.$$

Φ OF THE FOURTH DEGREE.

27. *Whenever Φ is homogeneous in u and z, the variables may be separated by the substitution,*

$$u = \lambda u' + \mu z',$$
$$z = \lambda' u' + \mu' z',$$

provided that the invariant T of a binary quartic vanishes. For, let

$$\Phi \equiv au^4 + 4bu^3z + 6cu^2z^2 + 4duz^3 + ez^4;$$

then b, c, d must $= 0$.

$$\therefore T = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0.$$

In general, we cannot by the substitution,

$$u = \lambda u' + \mu z' + \nu,$$
$$z = \lambda' u' + \mu' z' + \nu',$$

separate u and z when Φ is of the fourth degree. For there are six terms which must disappear, and but four effective constants at our disposal in the matrix of transformation. Hence there exist two conditions among the coefficients whenever separation is possible. Let

$$\begin{aligned}\Phi \equiv\ & au^4 + bz^4 + cv^4 \\ & + 6fz^2v^2 + 6gv^2u^2 + 6hu^2z^2 + 12lu^2zv + 12mz^2vu + 12nv^2uz \\ & + 4a_2u^3z + 4a_3u^3v + 4b_1z^3u + 4b_3z^3v + 4c_1v^3u + 4c_2v^3z.\end{aligned}$$

To have the required form, obviously h, l, m, n, b_1, a_2 must each be zero. The sextinvariant of a ternary quartic is

$$\begin{vmatrix} a, & h, & g, & l, & a_3, & a_2 \\ h, & b, & f, & b_3, & m, & b_1 \\ g, & f, & c, & c_2, & c_1, & n \\ l, & b_3, & c_2, & f, & n, & m \\ a_3, & m, & c_1, & n, & g, & l \\ a_2, & b_1, & n, & m, & l, & h \end{vmatrix}.$$

But the last row becomes zero. Hence, *whenever u and z can be separated by the above substitution, the sextinvariant vanishes.*

The converse of this statement is not in general true, because there is, as we have seen, one more condition to be satisfied. This may not be as simple as the one just found.

Of course we can deprive Φ of the six terms whose coefficients are a_2, b_1, etc., by means of a homographic substitution; but that would be less convenient for the present purpose.

GENERAL CASE, Φ OF THE FORM $\omega(u, z) + \psi(z)$ OR $\chi(u) + \omega(u, z)$.

28. If ω be of a degree not higher than the fourth, (in u for the first case, in z for the second), it is possible to separate u and z by introducing irrational expressions. But owing to the complicated results, the degree of ω is practically limited to the second in the general case. There are a few special cases where separation is possible; e. g.

$$u^{2m} + \xi u^m = \zeta.$$

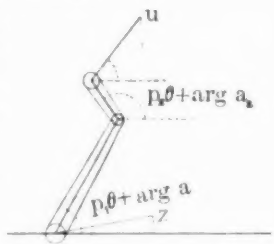
where ξ, ζ are any functions of z; and so for $z^{2n}+$, etc.

29. The curves $I^{(\rho = k)}_{\phi(z)}$ admit of a simple organic description when $\varphi(z)$ is of the form

$$\Sigma a z^{p},$$

where a is general, and p is any real number, entire or fractional.

Imagine a series of bars whose lengths are ρ^{p_1} mod. a_1, ρ^{p_2} mod. a_2, . . . , and suppose them jointed together and moved with angular velocities proportional to p_1, p_2, . . . , starting with angles of inclination to the x-axis equal to arg. a_1, arg. a_2, ...



The constant term of φ may be regarded as a bar without motion.

It does not seem practicable to describe the orthogonal set, $I^{(\theta = a)}_{\phi(z)}$, by any similar mechanism.

APPROXIMATE EQUATIONS OF IMAGES.

30. Let $u = \varphi(z)$, where φ is any real function, algebraic or transcendental. Now if z move in a region where $\mathfrak{r}\varphi$ and $\mathfrak{p}\varphi$ are convergent, we may obtain approximate expressions for $I^{(x = a)}_{\phi(z)}$ or $I^{(y = \beta)}_{\phi(z)}$ *whose degree shall be equal to the highest power of the y or x used in the development.*

Let $x = a$; then we have

$$X = a_0 + a_2 y^2 + a_4 y^4 + \ . \ . \ . \ ,$$
$$Y = a_1' y + a_3' y^3 + a_5' y^5 + \ . \ . \ . \ ;$$

$$\therefore\ I^{(x = a)}_{\phi_2} = \begin{vmatrix} a_2, & 0, & a_0 - X \\ a_1', & -Y, & 0 \\ 0, & a_1', & -Y \end{vmatrix} = 0.$$

The subscript of φ indicates the highest power of y taken.

$$I^{(x = a)}_{\phi_3} = \begin{vmatrix} a_2, & 0, & a_0 - X, & 0, & 0 \\ 0, & a_2, & 0, & a_0 - X, & 0 \\ 0, & 0, & a_2, & 0, & a_0 - X \\ a_3', & 0, & a_1', & -Y, & 0 \\ 0, & a_3', & 0, & a_1', & -Y \end{vmatrix} = 0.$$

The general equation takes one of the two forms:

(1.) *n odd*

$$I^{x = a}_{\phi_n} = X_0 Y^{n-1} + X_2 Y^{n-3} + X_4 Y^{n-5} + \ . \ . \ . \ + X_{n-2} Y^2 + X_n Y^0 = 0,$$

(2.) *n even*

$$I_{\phi_n}^{x=a} = X_0 Y^n + X_1 Y^{n-2} + X_3 Y^{n-4} + \ .\ .\ .\ + X_{n-3} Y^2 + X_{n-1} Y^0 = 0,$$

where X_r is a polynomial of the rth degree in X.

Since $x = a$ is a path symmetric with reference to the x-axis, it follows from Art. 13 that I is symmetric with reference to the X-axis. In other words, Y can enter by even powers only.

In the other system $y = \beta$, we develop X and Y in powers of x; we then have

$$b_0 + b_2 x^2 + b_4 x^4 + \ .\ .\ .\ = \underset{\phi \text{ even}}{X}, \quad = \underset{\phi \text{ odd}}{Y},$$

$$b_1' x + b_3' x^3 + b_5' x^5 + \ .\ .\ .\ = \underset{\phi \text{ even}}{Y}, \quad = \underset{\phi \text{ odd}}{X}.$$

When φ is an even function, $I_\phi^{(y-\beta)}$ is of the same form as I_ϕ^{x-a}; if an odd function, it will be of the same form after interchanging X and Y.

LINES OF EQUAL EXPANSION AND THEIR ORTHOGONAL TRAJECTORIES.

31. If each element along the u-path bears a constant ratio to the corresponding element of the z-path, this latter is called a *line of equal expansion*.

Let dA, da represent elementary u- and z-areas, and let accents denote differentiation. Then

$$dA = (\text{mod. } du)^2, \quad da = (\text{mod. } dz)^2.$$

Let $x = a$ or $y = \beta$; then mod. $dz = \sqrt{[(dx)^2 + (dy)^2]}$ becomes dy or dx, mod. du becomes $\sqrt{(X'^2 + X_{,}^2)}\,.\,dy,\ dx$;

$$\therefore \frac{dA}{da} = X'^2 + X_{,}^2 = k,$$

is the equation of the system of lines of equal expansion where k is the parameter.

The system $\dfrac{X'}{X_{,}} = c$ *cuts the system* $X'^2 + X_{,}^2 = k$ *orthogonally.*

For, differentiating the former, we have

$$\frac{dy}{dx} = -\frac{X''X_{,} - X'X_{,}'}{X_{,}'X_{,} - X'X_{,,}} = \frac{X'X_{,}' + X_{,}X_{,,}}{X'X'' + X_{,}X_{,}'}, \tag{1}$$

since $X'' = -X_{,,}$. The latter system gives

$$\frac{dy}{dx} = -\frac{X'X'' + X_{,}X_{,}'}{X'X_{,}' + X_{,}X_{,,}}. \tag{2}$$

But (1) is the reciprocal of (2) with its sign changed.

FOCI.

32. A *focus* of a function u is a point in the u-plane in whose immediate vicinity the transformed elements are infinitely small in comparison with their for-

mer magnitude. The corresponding point in the z-plane may be called an *anti-focus*.

To find the anti-foci of a given function.

We have, by definition,

$$\frac{\text{mod.}\, du}{\text{mod.}\, dz} = 0.$$

$$\therefore \sqrt{(X'^2 + Y'^2)} = 0; \text{ i. e. } X' = i\,Y', = -\,i\,X_,.$$

This equation can be true only when X' and $X_,$ are separately equal to zero, since these functions are by hypothesis both real:

$$\therefore \begin{cases} X' = 0, \\ X_, = 0, \end{cases}$$

determine the anti-foci.

TRANSCENDENTAL FUNCTIONS.

33. When $u = e^z$,

then
$$X = e^x \cos y,$$
$$Y = e^x \sin y.$$

If $x = \alpha$, we have $X^2 + Y^2 = e^{2\alpha}$; i. e. u describes a circle whose centre is at the origin and whose radius is equal to e^α.

If $y = \beta$, then $Y = X \tan \beta$; i. e. u describes a straight line through the origin making an angle β with the X-axis.

If $f(x, y) = 0$ be the equation of the path, then that of the image is

$$f\left[\tfrac{1}{2}\log\,(X^2 + Y^2), \tan^{-1}\frac{Y}{X}\right] = 0.$$

34. When $u = \sin z$,

$$X = \sin x \cosh y,$$
$$Y = \cos x \sinh y.$$

Let $x = \alpha$, and let $a = \cos \alpha$, $b = \sin \alpha$;

then
$$\frac{X^2}{b^2} - \frac{Y^2}{a^2} = 1.$$

Next, let $y = \beta$, and let $a' = \cosh \beta$, $b' = \sinh \beta$;

then
$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1.$$

When $u = \cos z$,

$$X = \cos x \cosh y,$$
$$Y = -\sin x \sinh y;$$
$$\therefore \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1,$$
$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1.$$

The ellipses are the same as before; but the hyperbolas have their axes interchanged. All the ellipses and hyperbolas are bi-confocal.

35. $u = \operatorname{sn} z$.

(1.) First of all, let us examine the region about the focus $X = 1$, $Y = 0$ whose anti-focus is $x = K$, $y = 0$.

Let $z = K + z'$; then

$$u = \operatorname{sn}(K + z') = \frac{\operatorname{cn} z'}{\operatorname{dn} z'} = \frac{1 - \frac{1}{2}z'^2 + \ldots}{1 - \frac{1}{2}z'^2 . k^2 + \ldots} = 1 - \frac{1 - k^2}{2} z'^2 + \ldots$$

Neglecting powers of z' beyond the second, we have

$$X = 1 - \tfrac{1}{2}(1 - k^2)(x'^2 - y'^2).$$
$$Y = -(1 - k^2)\, x'\, y'.$$

Let $x' = \alpha$, and write X' for $X - 1$; then

$$-X' = \tfrac{1}{2}(1 - k^2)\left[\alpha^2 - \frac{Y^2}{(1 - k^2)^2\, \alpha^2}\right]$$

is the equation of $I_{\operatorname{sn} z}^{(x' = \alpha)}$, where the focus is the origin. For the system $y = \beta$, we have

$$-X' = \tfrac{1}{2}(1 - k^2)\left[\frac{Y^2}{(1 - k^2)\,\beta^2} - \beta^2\right].$$

In like manner, may be discussed the region about the focus $u = 1/k$ whose anti-focus is $x = K$, $y = K'$. In this case,

$$\operatorname{sn}(K + K'i + z') = \frac{1}{k}\frac{\operatorname{dn} z'}{\operatorname{cn} z'} = \frac{1}{k}\left(1 + \frac{1 - k^2}{2} z'^2 + \ldots\right).$$

If z be near $K'i$, and u near ∞, let $z = K'i + z'$; then

$$u = \operatorname{sn}(K'i + z') = \frac{1}{k \operatorname{sn} z'} \doteq \frac{1}{kz'};$$

z' being very small.

If z' describe the system $x' = -\alpha$, u will describe a system of circles whose centres lie upon the X-axis and which have a common point of contact at the origin. The radius of any particular circle will be equal to $1/2k\alpha$.

Now let $y' = -\beta$ in the equation

$$u = \frac{1}{kz'}.$$

These circles have their centres upon the Y-axis at distances equal to $1/2k\beta$ from the origin, where they have a common point of contact.

(2.) Curve of maximum abscissas.

$$u = \operatorname{sn}(x + iy) = \frac{\operatorname{sn} x \operatorname{cn} iy \operatorname{dn} iy + \operatorname{sn} iy \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy},$$

$$X = \frac{\operatorname{sn} x \operatorname{cn} iy \operatorname{dn} iy}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy},$$

$$\frac{\partial X}{\partial x} = \frac{\operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy\,(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy) + 2k^2 \operatorname{sn}^2 x \operatorname{cn} x \operatorname{dn} x \operatorname{sn}^2 iy \operatorname{cn} iy \operatorname{dn} iy}{(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy)^2}$$

$$= \frac{1 + k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy}{(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy)^2} \operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy.$$

$\partial X/\partial x$ may be made equal to zero by equating to zero, either

$$\operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy, \quad \text{or} \quad 1 + k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy$$

y, the parameter, being less than K'.

If $$x = K, \quad \operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy = 0;$$

therefore that part of the X-axis which joins the foci 1 and $1/k$ is the locus of the points of contact of a tangent, parallel to the Y-axis, with the system

$$I^{(y=\beta)}_{\operatorname{sn} z}.$$

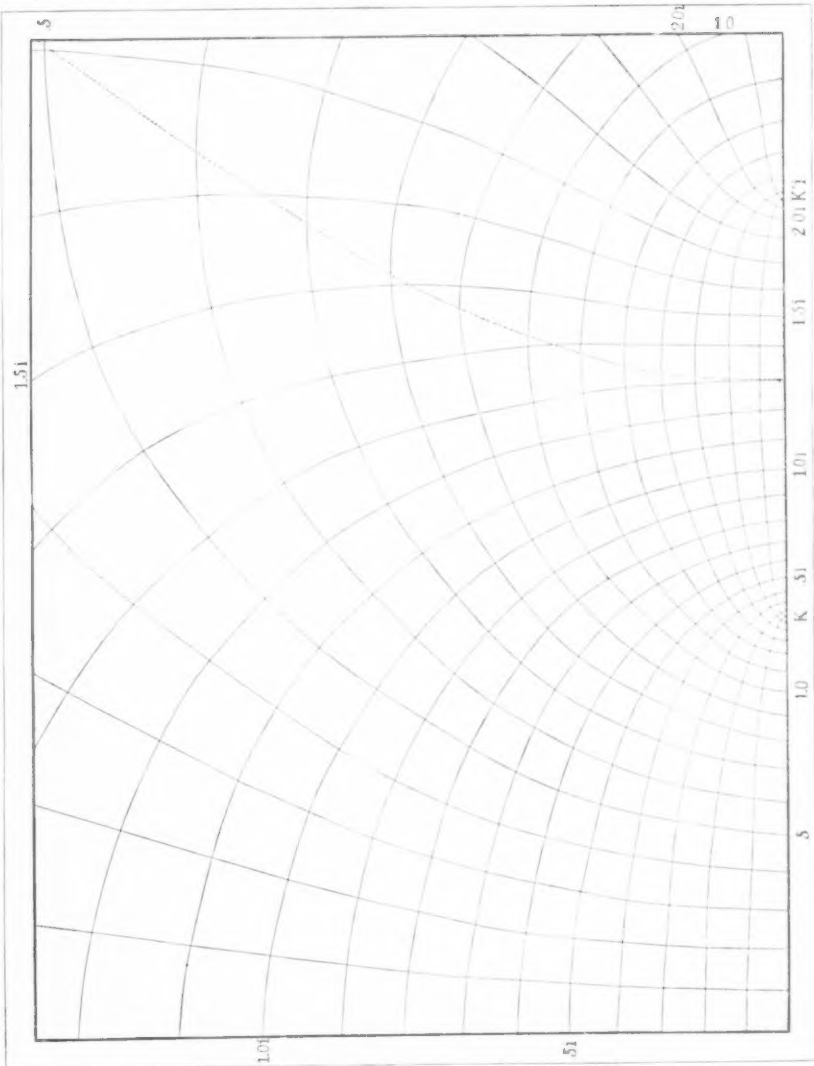
The other locus of vertical tangents is determined by aid of the equation

$$1 + k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy = 0, \quad \text{or} \quad \operatorname{sn} x \operatorname{sn} iy = \pm i/k.$$

It was for this locus that the above name was intended. Obviously the curve of maximum abscissas for $I^{(y=\beta)}_{\operatorname{sn} z}$ is the curve of maximum ordinates for $I^{(x=\alpha)}_{\operatorname{sn} z}$.

(3.) Example. Suppose $k = \frac{1}{2}$; and let the region of the z-plane considered, be a rectangle whose edges are K and K' in length.

For this value of k, $K = 1.68575$ and $K' = 2.1565$. By means of expansions and addition formulae, we construct the following tables:—


.5
2.0i
1.0
2.0i K'
1.5i
1.0i
.5i
K
1.0
.5
1.5i
1.0i
.5i

z	sn z	cn z	dn z.	z	$-i$ sn iz	cn iz	dn iz
.1	.09979	.99501	.99875	.1	.10021	1.00501	1.00125
.2	.19835	.98013	.99507	.2	.20168	1.02013	1.00507
.3	.29447	.95566	.98910	.3	.30572	1.04569	1.01162
.4	.38704	.92206	.98110	.4	.41374	1.08221	1.02117
.5	.47508	.87994	.97138	.5	.52730	1.13051	1.03417
.6	.55773	.83002	.96033	.6	.64822	1.19171	1.05121
.7	.63429	.77310	.94839	.7	.77864	1.26739	1.07311
.8	.70421	.70999	.93596	.8	.92121	1.35964	1.10098
.9	.76708	.64155	.92352	.9	1.07929	1.47135	1.13632
1.0	.82263	.56857	.91149	1.0	1.25728	1.60647	1.18118
1.1	.87069	.49183	.90026				
1.2	.91117	.41203	.89019	1.5	2.77546	2.95011	1.71050
1.3	.94405	.32981	.88159				
1.4	.96934	.24572	.87470	2.0	12.7131	12 7523	6.43471
1.5	.98706	.16039	.86973				
1.6	.99724	.07422	.86682				
K	1.00000	.00000	.86603	K'	∞	∞	∞

Suppose a sufficient number of real and purely imaginary values of z to have been taken, and the values of sn z, cn z, and dn z computed and thus tabulated; then, by the addition formulae, we can find any mixed value of

$$u\ (= \text{sn}\ z,\ \text{cn}\ z,\ \text{or}\ \text{dn}\ z.)$$

The dotted line upon the chart is the curve of maximum abscissas for the system $I_{\left(\frac{y}{\text{sn}\,z}=\beta\right)}$. To find where this curve intersects the X-axis, make $x = K$; $\therefore$ sn $x = 1$; sn x sn $iy = 2i$ becomes $-i$ sn $iy = 2$. Had we a complete table, we should find y equal to about 1.3. Therefore this curve meets the X-axis near the point where the curve $I_{\left(\frac{y}{\text{sn}\,z}=1.3\right)}$ meets it.

About $\frac{1}{6}$ of the part of the z-plane considered, lies, when transformed, beyond the limits of our chart.

By making use of the same computations we could, of course, construct charts for cn z, dn z, sn$^{-1} z$, etc.

SOLID IMAGES.

36. Let the right versor i operate in planes parallel to xy, and j, in planes parallel to xz, then the affix of any point in space is

$$u = x + iy + jz.$$

Let
$$U = \varphi(u) = X + iY + jZ.$$

By Taylor's Theorem,

$$U = \varphi(x) + \frac{iy + jz}{1} \frac{\partial \varphi(x)}{\partial x} + \frac{(iy + jz)^2}{2!} \frac{\partial^2 \varphi(x)}{\partial x^2} + \ldots;$$

whence, expanding the binomials $(iy + jz)^r$, and omitting all terms of the form $Ki^m j^n y^m z^n$ where m and n are both odd, we have

$$\begin{aligned} X = \varphi(x) - \frac{y^2 + z^2}{2!} \varphi''(x) + \frac{y^4 + 6y^2z^2 + z^4}{4!} \varphi^{\text{iv}}(x) \\ - \frac{y^6 + 15y^4z^2 + 15y^2z^4 + z^6}{6!} \varphi^{\text{vi}} + \ldots, \end{aligned} \tag{1}$$

$$Y = y\varphi'(x) - \frac{y^3 + 3yz^2}{3!} \varphi'''(x) + \frac{y^5 + 10y^3z^2 + 5yz^4}{5!} \varphi^{\text{v}}(x) - \ldots, \tag{2}$$

$$Z = z\varphi'(x) - \frac{z^3 + 3zy^2}{3!} \varphi'''(x) + \frac{z^5 + 10z^3y^2 + 5zy^4}{5!} \varphi^{\text{v}}(x) - \ldots. \tag{3}$$

It is easily shown, from (1), (2), (3), when z is constant, that

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y},$$

$$\frac{\partial Y}{\partial x} = -\frac{\partial X}{\partial y};$$

and when y is constant, that

$$\frac{\partial X}{\partial x} = \frac{\partial Z}{\partial z},$$

$$\frac{\partial Z}{\partial x} = -\frac{\partial X}{\partial z}.$$

Let us return for a moment to the consideration of plane images.

(a) If $u = x + iy + jz$ move along any two intersecting curves whose plane is parallel to xy, then the projections of their images upon XY intersect at the same angles as did the original paths of the variable u. For we have already seen that if

$$U = X + iY,$$

and if z be constant, U is a function of $u' = x + iy$. In like manner, if

$$U = X + jZ,$$

and if y be constant, U is a function of $u'' = x + jz$. Therefore, in either case, all angles are preserved in the projections (upon the plane XY or XZ).

Let
$$X = \varphi_1 (x, y, z), \tag{1'}$$
$$Y = \varphi_2 (x, y, z), \tag{2'}$$
$$Z = \varphi_3 (x, y, z); \tag{3'}$$

also, let the equations of the path of u be $f_1 (x, y, z) = 0$, $f_2 (x, y, z) = 0$. If we eliminate x, y, z, the two resulting equations in X, Y, Z determine a curve in space of three dimensions—the path of U.

We shall next discuss the infinitesimal elements of the U-space.

If V be any function ψ of x, y, z, and $V_1 = \psi (x_1, y_1, z_1)$, $V_2 = \psi (x_2, y_2, z_2)$ be two special values where x_1, y_1, z_1; x_2, y_2, z_2 are points very near together; then, if x_1, y_1, z_1 become $x_1 + h$, $y_1 + k$, $z_1 + l$, and if x_2, y_2, z_2 become $x_2 + h$, $y_2 + k$, $z_2 + l$, respectively, V_1 and V_2 are increased by equal amounts as $h, k, l \doteq 0$.
For

$$V_1 + H_1 = \psi (x_1 + h,\ y_1 + k,\ z_1 + l) = V_1 + h \frac{\partial V_1}{\partial x_1} + k \frac{\partial V_1}{\partial y_1} + l \frac{\partial V_1}{\partial z_1} + \ldots,$$
$$V_2 + H_2 = \psi (x_2 + h,\ y_2 + k,\ z_2 + l) = V_2 + h \frac{\partial V_2}{\partial x_2} + k \frac{\partial V_2}{\partial y_2} + l \frac{\partial V_2}{\partial z_2} + \ldots.$$

Now $H_1 \doteq H_2$; for, as $x_1 \doteq x_2$, $y_1 \doteq y_2$, $z_1 \doteq z_2$;

$$\frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2} = \varepsilon,$$

where ε is an infinitesimal of the first order;

$$\therefore h \left(\frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2} \right) = h\varepsilon,$$

an infinitesimal of the second order. So proceed with the quantities multiplied by k and l;

$$\therefore H_1 - H_2 \doteq 0.$$

Let x_1, y_1, z_1; x_2, y_2, z_2 be the co-ordinates of the extremities of a very short; and we may suppose, straight line; then, of course, $x_1 + h$, $y_1 + k$, $z_1 + l$, $x_2 + h$, $y_2 + k$, $z_2 + l$ determine an equal and parallel line. Suppose

$$X_1 = \psi_1 (x_1, y_1, z_1),$$
$$Y_1 = \psi_2 (x_1, y_1, z_1),$$
$$Z_1 = \psi_3 (x_1, y_1, z_1),$$
$$X_2 = \psi_1 (x_2, y_2, z_2),$$
$$\ldots \qquad \ldots$$

to be co-ordinates of the extremities of a very short line; then, if the line x_1, y_1, z_1; x_2, y_2, z_2 move parallel to itself, so does the line

$$X_1, Y_1, Z_1;\ X_2, Y_2, Z_2.$$

For, giving to x, y, z the necessary increment h, k, l, we have

$$\begin{aligned} &X_1 + H \text{ for } X_1, \\ &X_2 + H \quad \text{"} \quad X_2, \\ &Y_1 + K \quad \text{"} \quad Y_1, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \ \cdot\ \cdot\ \cdot \end{aligned}$$

(*b*) Hence, if the u-space be divided into elementary parallelopipeds, so is the U-space, regardless of the nature of the function ψ.

Therefore, if the u-space be divided into elementary cubes by planes parallel to the co-ordinate planes, the transformed elements will be parallelopipeds, the projections of the XY and XZ faces of which, upon the co-ordinate planes XY, XZ, respectively, will be perfect squares. (*a*), (*b*).

EXAMPLES.

(1) If
$$\begin{aligned} U &= u^2, \\ X &= x^2 - y^2 - z^2, \\ Y &= 2xy, \\ Z &= 2xz. \end{aligned}$$

(2) If
$$\begin{aligned} U &= e^u, \\ X &= e^x \cos y \cos z, \\ Y &= e^x \sin y \cos z, \\ Z &= e^x \cos y \sin z. \end{aligned}$$

(3) If
$$\begin{aligned} U &= \sin u, \\ X &= \sin x \cosh y \cosh z, \\ Y &= \cos x \sinh y \cosh z, \\ Z &= \cos x \cosh y \sinh z. \end{aligned}$$

These are analogous to plane images, and are reduced to them by making $z = 0$.

ON THE EXPANSION OF sn x.

By Mr. Rollin A. Harris, Jamestown, N. Y.

(1.) Since $\operatorname{sn}(-x) = -\operatorname{sn} x$, we may assume

$$\operatorname{sn} x = A_0 x + A_1 x^3 + A_2 x^5 + \ldots;$$
$$\therefore \operatorname{sn} y = A_0 y + A_1 y^3 + A_2 y^5 + \ldots,$$
$$\operatorname{sn}(x+y) = A_0(x+y) + A_1(x+y)^3 + \ldots,$$
$$\operatorname{cn} x \operatorname{dn} x = \operatorname{sn}' x = A_0 + 3A_1 x^2 + 5A_2 x^4 + \ldots;$$

and so for $\operatorname{cn} y \operatorname{dn} y$.

Substituting these developments in the formula

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn}^2 x - \operatorname{sn}^2 y}{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y - \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x},$$

we have, after clearing of fractions,

$$\begin{aligned} &[A_0(x+y) + A_1(x+y)^3 + \ldots] \\ &\quad \times \left[\begin{array}{l} (A_0 x + A_1 x^3 + \ldots)(A_0 + 3A_1 y^2 + \ldots) \\ -(A_0 y + A_1 y^3 + \ldots)(A_0 + 3A_1 x^2 + \ldots) \end{array}\right] \\ &= (A_0 x + A_1 x^3 + \ldots)^2 - (A_0 y + A_1 y^3 + \ldots)^2. \end{aligned} \qquad (a)$$

Since the second member contains terms of the form cx^n or cy^n only, it follows that, if the first member be expanded and similar terms united, the coefficient of any resulting term, whose form is $cx^p y^q$ where $p, q > 0$, must be equal to zero. We could thus obtain relations between the coefficients $A_0, A_1, \ldots A_n$. But actual multiplication is not necessary, since we are not concerned with the entire product. We proceed as follows:—

(2.) Performing the multiplication indicated in the first term of the second factor of the first member of (a), we have

$$\begin{aligned} &(A_0 x + A_1 x^3 + \ldots)(A_0 + 3A_1 y^2 + \ldots) \\ &\qquad = A_0 A_0 y^0 x + A_0 A_1 y^0 x^3 + A_0 A_2 y^0 x^5 + \ldots \\ &\qquad + 3A_1 A_0 y^2 x + 3A_1 A_1 y^2 x^3 + 3A_1 A_2 y^2 x^5 + \ldots \\ &\qquad + 5A_2 A_0 y^4 x + 5A_2 A_1 y^4 x^3 + 5A_2 A_2 y^4 x^5 + \ldots \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

		Order in x.				
		1	3	5	7	
Order in y.	0	A_0A_0	A_0A_1	A_0A_2	A_0A_3	. . .
	2	$3A_1A_0$	$3A_1A_1$	$3A_1A_2$	$3A_1A_3$	. . .
	4	$5A_2A_0$	$5A_2A_1$	$5A_2A_2$	$5A_2A_3$	. . .
	6	$7A_3A_0$	$7A_3A_1$	$7A_3A_2$	$7A_3A_3$	. . .
		. . .	. . .	. . .	. . .	. . .

= (table above) (β)

(3.) Because of symmetry, if x and y of this table be interchanged, we have the product of $A_0y + A_1y^3 + \ldots$ and $A_0 + 3A_1x^2 + \ldots$.

(4.) The first factor of (α) may be written $\sum_{n=0}^{n=\infty} A_n(x+y)^{2n+1}$,

or (γ)

	0	1	2	3	4	5	6	7
0		A_0		A_1		A_2		A_3
1	A_0		$3A_1$		$5A_2$		$7A_3$	
2		$3A_1$		$10A_2$		$21A_3$		
3	A_1		$10A_2$		$35A_3$			
4		$5A_2$		. . .				
5	A_2		. . .					
6		. . .						
7	. . .							

Suppose that we multiply (β) by (γ) and wish to find all terms in which x and y have the exponents p and q, respectively. Each of such terms will consist of two factors $c_{st}x^sy^t$, $c_{uv}x^uy^v$, the one from (β), the other from (γ), where $s + u = p$ and $t + v = q$. Since $s \leqq p$ and $t \leqq q$, we see at once, what factors can come from (β); and since $p = s + u$ and $q = t + v$, the factors from (γ) are determined. In fact, we have only to multiply the first term of (β) by that term of (γ) whose order

is the highest permissible; again, the next higher in (β) with reference to either variable by the next lower in (γ) with reference to the same variable, and so continue until s and t have reached their limits.

(5.) EXAMPLE. Let it be required to find the entire coefficient of x^5y^3.

From what has just been said, (β) multiplied by (γ) would give as the coeffient of x^5y^3

$$\begin{aligned} &A_0A_0 \,.\, 35A_3 + \ A_0A_1 \,.\, 10A_2 + \ A_0A_2 \,.\, A_1 \\ &+ 3A_1A_0 \,.\, \ 5A_2 + 3A_1A_1 \,.\, \ 3A_1 + 3A_1A_2 \,.\, A_0. \end{aligned}$$

Now interchange x and y; there results as the coefficient of y^5x^3

$$\begin{aligned} &A_0A_0 \,.\, 21A_3 + \ A_0A_1 \,.\, A_2 \\ &+ 3A_1A_0 \,.\, 10A_2 + 3A_1A_1 \,.\, A_1 \\ &+ 5A_2A_0 \,.\, \ 3A_1 + 5A_2A_1 \,.\, A_0. \end{aligned}$$

But these must be equal, since the entire coefficient of x^5y^3 equals zero [(3), (4)]. Collecting and transposing, we have

$$7A_3A_0^2 - 11A_2A_1A_0 + 3A_1^3 = 0.$$

From other considerations, we know that $A_0 = 1$;

$$\therefore\ 7A_3 - 11A_2A_1 + 3A_1^3 = 0.$$

(6.) From (γ), it is evident that we can always bring A_m into an equation involving A's of lower subscript only, if we take $p + q = 2(m + 1)$. For example, we could have obtained the above equation by using x^6y^2 instead of x^5y^3. Obviously, the smaller we can take q, the more simple will be the operation.

If $q = 1$, the result is an identity, and so of no value to us; but if $q = 2$, the result in general will not be an identity. We observe that the coefficient of the A_m, $q = 2$, from (γ) is the same as the coefficient of the third term of $(a + b)^{2m+1}$, or $m(2m + 1)$. In (β), A_m does not occur while we are considering x^py^2. Now interchange x and y; i. e. consider x^2y^p. The A_m from (γ) has the same coefficient as the second term of $(a + b)^{2m+1}$, or $2m + 1$; from (β), it also has the coefficient $2m + 1$. These added together give $2(2m + 1)$; but $2(2m + 1)$ is not equal to $m(2m + 1)$, except for $m = 2$. Since, in general, we can obtain the required relations by making $q = 2$, we shall hereafter confine ourselves to that value. The operation then becomes very simple. E. g. Required the value of A_4.

Here $p + q = 10$; therefore $p = 8$. We have, then,

$$\begin{aligned} &A_0(A_0 \,.\, 36A_4 + \ A_1 \,.\, 21A_3 + \ A_2 \,.\, 10A_2 + \ A_3 \,.\, 3A_1) \\ &+ 3A_1(A_0 \,.\, \ A_3 + \ A_1 \,.\, \ A_2 + \ A_2 \,.\, \ A_1 + \ A_3 \,.\, \ A_0) \\ &= A_0(A_0 \,.\, \ 9A_4 + 3A_1 \,.\, \ 7A_3 + 5A_2 \,.\, \ 5A_2 + 7A_3 \,.\, 3A_1 + 9A_4 \,.\, A_0); \end{aligned}$$

$$\therefore\ 6A_4 - 4A_3A_1 - 5A_2^2 + 2A_2A_1^2 = 0.$$

(7.) The relation sought is embodied in the following general equation:—

$$A_0 \sum_{r=1}^{r=m} r(2r+1) A_r A_{m-r} + 3A_1 \sum_{r=1}^{r=m} A_{r-1} A_{m-r}$$
$$= A_0 \sum_{r=1}^{r=m+1} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1};$$

$$\therefore\ A_m = \frac{\left\{\begin{array}{l} A_0 \cdot \displaystyle\sum_{r=2}^{r=m+1} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1} \\ -\,3A_1 \displaystyle\sum_{r=1}^{r=m} A_{r-1} A_{m-r} - A_0 \displaystyle\sum_{r=1}^{r=m-1} r(2r+1) A_r A_{m-r} \end{array}\right\}}{(m-2)(2m+1) A_0^2}.$$

m odd

$$A_m = \frac{\left\{\begin{array}{l} 2A_0 \displaystyle\sum_{r=2}^{r=\frac{1}{2}(m+1)} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1} \\ -\,6A_1 \displaystyle\sum_{r=1}^{r=\frac{1}{2}(m-1)} A_{r-1} A_{m-r} - 3A_1 A^2_{\frac{1}{2}(m-1)} - \displaystyle\sum_{r=1}^{r=m-1} r(2r+1) A_r A_{m-r} \end{array}\right\}}{(m-2)(2m+1) A_0^2}.$$

m even

$$A_m = \frac{\left\{\begin{array}{l} 2A_0 \displaystyle\sum_{r=2}^{r=\frac{1}{2}m} (2r-1)(2m-2r+3) A_{r-1} A_{m-r+1} + (m+1)^2 A_0 A^2_{\frac{1}{2}m} \\ -\,6A_1 \displaystyle\sum_{r=1}^{r=\frac{1}{2}m} A_{r-1} A_{m-r} - A_0 \displaystyle\sum_{r=1}^{r=m-1} r(2r+1) A_r A_{m-r} \end{array}\right\}}{(m-2)(2m+1) A_0^2}.$$

By giving m the values 3, 4, . . . , we have

$$\begin{aligned} &7A_3 - 11A_2A_1 + 3A_1^3 = 0, \\ &6A_4 - 4A_3A_1 - 5A_2^2 + 2A_2A_1^2 = 0, \\ &11A_5 - 3A_4A_1 - 13A_3A_2 + 2A_3A_1^2 + A_2^2A_1 = 0, \\ &26A_6 - A_5A_1 - 22A_4A_2 + 3A_4A_1^2 - 14A_3^2 + 3A_3A_2A_1 = 0. \end{aligned}$$

. . . .

OBSERVATIONS.

All terms are of order three, counting the A_0's, and the weight of each term of any of these equations equals the greatest subscript, equals the number of terms.

A_1 and A_2 are supposed to be known; they may be called the *source* of all succeeding ones. For in terms of these two, any that follow may be expressed.

NOTE.—A similar method is applicable to cn x and dn x.

ON A PROPERTY OF AN IMAGINARY LINE PASSING THROUGH ONE OF THE CIRCULAR POINTS AT INFINITY.

By Mr. James McMahon, Ithaca, N. Y.

In the *Messenger of Mathematics*, Vol. XVI. p. 2, appears the important statement that it would be more correct to say that the line $y = ix$ makes an indeterminate angle with itself, than to say (as is commonly done) that it is perpendicular to itself. The analytical reason given is that $\frac{m - m'}{1 + mm'}$ takes the form 0/0 when $m = m' = i$. This reason does not seem quite conclusive.

I think, for analytical purposes, it would be more satisfactory to state the the principle as follows:—

If two variable lines make any constant angle φ (real or imaginary) with each other, and if the equation of one of them approach the limiting form $y - y_0 = ix$, so will the equation of the other (unless $\tan^2 \varphi = -1$).

To prove this, let $m = (1 + a)i$, and $m' = (1 + a')i$;

then
$$\tan \varphi = \frac{a - a'}{a + a' + aa'}\, i, = k, \text{ suppose};$$

$$\therefore \frac{a'}{a} = \frac{i - k}{i + (1 + a)k} \doteq \frac{i - k}{i + k}, \text{ when } a \doteq 0;$$

therefore, when k^2 is any constant except -1, a and $a' \doteq 0$ together, and the equation of each line approaches the limiting form stated above.

When a constant angle φ turns round a fixed point in a plane, its sides generate two homographic pencils, which also form a system in involution; and the above principle shows that when a ray of either pencil approaches the position in which it passes through either of the circular points at infinity, then will the corresponding ray of the other pencil approach the same position. These two limiting positions give the two double lines of the involution-system.

When $\tan^2 \varphi = -1$, the line that makes an angle φ with either of the lines $y = \pm ix$, may have any direction whatever.

ON THE CHORDS OF A PARABOLA AND GENERALLY OF A CONIC.

By PROF. F. AMODEO, Naples, Italy.

I propose to show that the properties of the focal chords of a parabola given by Prof. Graves in a recent number of the ANNALS*, are common to all the chords of a parabola.

Let PP' be any chord of a parabola, T its pole, N the point of intersection of the parallels through P and P' to the tangents TP', TP. The line TN is the diameter of the parabola conjugate to the chord PP'.

Each chord PN, $P'N$ divides the other in the ratio 1 : 3.

If M, M' be the other ends of the chords PN, $P'N$, the chord joining MM' is parallel to PP' and three times as long.

Let us join the point T with one of the points S of the chord PP', and let T' be the point of intersection of the line TS with the chord MM'.

The line TT' is divided by S in the ratio 1 : 4.

Revolving the chord PP' around S, the point T will describe a right line s the polar of S, the point T' will describe another right line s', parallel to s, and therefore the envelope of the chord MM' is a parabola tangent to the chord s' at its middle point, having in common with the original parabola the point at infinity, and the pole S.

If the point S moves on the chord PP', we shall have an infinite number of parabolas, all tangent to the original parabola at the point at infinity, and to the chord MM' parallel to PP'.

Only two parabolas are bitangent to the original parabola, and they are those that correspond to the positions P, P' of the point S.

This property of the chords of a parabola is a special case of the following property of the chords of any conic.

Let P, P', P'' be three points of a conic, p, p', p'' their tangents, and M, M' the projections, on the conic, from P, P' of the points $p'p''$, pp''. If P, P', p, p' vary, the chord PP' revolving around one of its points S, the envelope of the chord MM' is a conic tangent to the original conic at P'', and having in common with it the pole S.

*See Vol. III. p. 153.

CONSTRUCTION OF PERSPECTIVE PROJECTIONS.

By Prof. W. H. Echols, Rolla, Mo.

The following note was suggested by an article from Professor Thornton, under the same heading, in the Annals of Mathematics, Vol. I. No. 1, in which the formulae are given for the computation of the co-ordinates of the perspective of a point referred to the centre of the picture as origin, the prime vertical and horizon as axes of the ordinate and abscissa, respectively. In the article referred to, while diagrams are given which indicate how these co-ordinates may be obtained graphically, attention is more particularly called to the arithmetical solution by use of the formulae.

It is the object of the present paper to point out the graphical solution of the same problem, in which it will be seen that only one of the perspective co-ordinates (the abscissa) will be required, the construction being effected by a method of *mixed co-ordinates*.

The system of reference being primarily the same, I quote from that article: "I refer the points of the original to an orthogonal co-ordinate system of three planes. The x-plane is the perspective plane; the y-plane is the vertical plane at right angles thereto through the centre of projection; the z-plane is the horizontal plane through the centre of projection. The distances of P from these planes are x, y, z; the perspective is referred to the axes traced by the y- and z-planes on the x-plane; and the distances of π, the perspective of P, from these axes are η, ζ. The distance of the point of view from the perspective plane is d." The radius vector of P is r, and (ρ, θ) the polar co-ordinates of π.

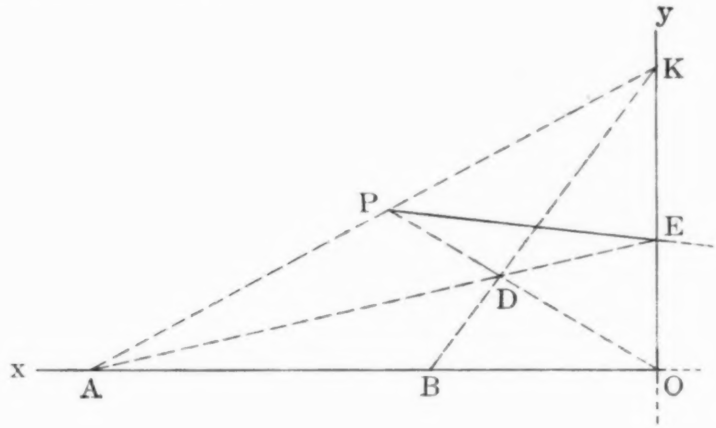
It is evident that ρ is the perspective of r, and that

$$\theta = \tan^{-1}\frac{z}{y}.$$

π is therefore located as the intersection of a line through the centre of the picture making an angle, $\tan^{-1} z/y$, with the horizon, with the vertical through $(\eta, 0)$.

In the actual construction of the perspective, no more data are needed than are required by Adhemar's Method of Scales. The plan, elevation, and perspective may be, and are best, on separate sheets of paper. On the plan, the x and y axes are located at pleasure. In elevation, the horizon is drawn in that position which produces the desired effect, or the actual elevation may be dispensed with under the same conditions as with the Method of Scales. The values of y and z are taken directly from the plan and elevation, or its equivalent, with the dividers, and the direction of ρ located in the picture by drawing *one line*. In the plan, if the point of view (C) is in reach, the line CP' (P' being the plan of any point

P) cuts off η on the y-axis, which applied to the perspective gives the position of π by *drawing* one line. If the point of view is not in reach, the following construction gives η with certainly the ease and accuracy of the vanishing point and reduced distance method, with the advantage of the auxiliary construction being on the plan and not in the picture.



In the diagram, C is on the x-axis at a distance d from O, and on the opposite side of Oy from the plan. Lay off on Ox in the direction opposite to OC the distances $OB = \frac{d}{1 + 2n}$, $OA = \frac{d}{n}$; giving to n any convenient value. The four points A, B, O, and C form an *harmonic range* and are fixed throughout the construction. If P' be any point in the plan, through P' draw AK, join KB draw OP', and through D draw AE. Then $P'E$ and the x-axis are concurrent in C, and $OE = \eta$. (*Venable's Modern Geometry*, p. 392).

If it be preferred not to locate such constructive points as K on the y-axis, then any other line through O will answer the same purpose. The value of n must be so selected as to give the positions of A and B to best suit the plan of the drawing.

In Adhemar's Method of Scales, in the perspective drawing the intersections near the horizon are oblique, here they are oblique when the points to be determined are near the prime vertical; in which case, for particular points, the values of ζ may be found instead of η by using the plan and making the same construction for the point (x, z), Oy being used as axis of z. Such points are frequently located by the intersection of the vector with some previously constructed line.

If C is within easy reach, both η and ζ may be taken from the plan and elevation as shown in Professor Thornton's diagrams, by simply drawing the two

lines which cut off these values on the axes, or otherwise by the quadrilateral construction, if C be far away, provided the elevation is such a projection as would give the desired scenic effect for the point of view fixed on the x-axis. Unfortunately for so simple a construction, elevations are usually such projections as prohibit the selection of the point of view to best advantage, whereas in the present construction, the position of the point of sight is independent of the form of the orthogonal projection in which the elevation is given, as only the *distances* of its points from the horizontal plane are required. This permits the selection of the position of C and the direction of the central ray with reference to the plan alone, and practically gives power to select any point of view we choose to take. The drawing in the perspective is as condensed as could be desired, and is practically confined to the same space for all distances of C.

SOLUTION OF EXERCISES.

165

In any triangle ABC let a circle be inscribed touching the sides AB, BC, CA in N, L, M respectively. Let the centre O of this circle be joined to the vertices and from O let OP, OQ be drawn perpendicular respectively to OC, OB, and cutting BC in P and Q. Then if NP and AQ be drawn, these lines will be parallel, as will also AP and MQ. [*F. H. Loud.*]

SOLUTION.

$$\angle BOC = 90^\circ + \tfrac{1}{2}A, \text{ and } \angle COP = 90^\circ;$$

$$\therefore \angle BOP = \tfrac{1}{2}A.$$

Also $$\angle OPB = 90^\circ + \tfrac{1}{2}C = \angle AOB.$$

Hence the triangles AOB and OPB are similar, and we have

$$\overline{BO}^2 = AB \cdot BP. \qquad (1)$$

Since $\frac{1}{2}B$ is a common angle in NBO and OBQ, these triangles are similar, and we have

$$\overline{BO}^2 = NB \cdot BQ. \qquad (2)$$

From (1) and (2) $\dfrac{AB}{NB} = \dfrac{BQ}{BP}$; $\therefore$ NP is parallel to AQ, which was to be proved.

[*W. O. Whitescarver.*]

168

If the normals at four points on a rectangular hyperbola meet in a point and the sum of the squares on the six distances between the four points, taken two together, is constant ($= k^2$), prove that the locus of the point of concourse of the normals is a circle. [*R. H. Graves.*]

SOLUTION I.

Let $x^2 - y^2 = a^2$ be the equation to the rectangular hyperbola, and (x, y) be the point of concourse of the normals. The co-ordinates of the four points are the roots of the biquadratics in x' and y'.

$$4x'^4 - 4xx'^3 + x'^2(x^2 - y^2 - 4a^2) + 4a^2xx' - a^2x^2 = 0, \qquad (1)$$

$$4y'^4 - 4yy'^3 - y'^2(x^2 - y^2 - 4a^2) - 4a^2yy' + a^2y^2 = 0. \qquad (2)$$

The sum of the squares of the differences of the roots of (1) is

$$3x^2 - 2(x^2 - y^2 - 4a^2).$$

The same function of the roots of (2) is

$$3y^2 + 2(x^2 - y^2 - 4a^2).$$

Hence the equation to the locus of the point of concourse is

$$3(x^2 + y^2) = k^2. \qquad [R. H. Graves.]$$

SOLUTION II.

Let $XY = c^2$ be the equation to the curve; $(X - x)X + (Y + y)X = 0$ is the equation to the normal at the point XY.

Eliminating Y and X alternately by means of the given relation, we have

$$X^4 - xX^3 - c^2yX + c^4 = 0,$$

and
$$Y^4 - yY^3 - c^2xY + c^4 = 0.$$

Also,
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 + (x_3 - x_4)^2 + (y_3 - y_4)^2 + (x_4 - x_1)^2 + (y_4 - y_1)^2$$
$$\equiv 3\Sigma x_1^2 + 3\Sigma y_1^2 - 2\Sigma x_1x_2 - 2\Sigma y_1y_2 = k^2,$$

and from the theory of equations,

$$\Sigma x_1 = x, \quad \Sigma y_1 = y, \quad \Sigma x_1x_2 = \Sigma y_1y_2 = 0.$$

x_1, x_2, x_3, x_4 being the four values of X, and y_1, y_2, y_3, y_4 the four values of Y, these conditions give

$$3\Sigma x_1^2 = x^2, \quad 3\Sigma y_1^2 = y^2.$$

Hence $3(x^2 + y^2) = k^2$, which is a circle; x and y being on each of the normals, and therefore on their point of concurse. . [*E. Frisby.*]

169

FIND the locus of the instantaneous centre of a tangent to an ellipse when one point of the tangent moves in the auxiliary circle. [*R. H. Graves.*]

SOLUTION.

Find the locus of the intersection of the normal and the perpendicular on it from the focus; i. e. eliminate φ from

$$\frac{ax}{\cos\varphi} - \frac{by}{\sin\varphi} = a^2 - b^2,$$

and
$$\frac{b(x - ae)}{\sin\varphi} + \frac{ay}{\cos\varphi} = 0.$$

This gives $(a^2 + b^2)\left(\frac{y}{x - ae} + \frac{x}{y}\right)^2 [y^2 + (x - ae)^2] = (a^2 - b^2)^2;$

or
$$(a^2 + b^2)\left(\frac{y}{x} + \frac{x + ae}{y}\right)^2 (x^2 + y^2) = (a^2 - b^2)^2,$$

if the origin is moved to the point $(ae, 0)$. [*R. H. Graves.*]

178

FIND the point in which the normal to $xy = m^2$ cuts the curve again.

SOLUTION.

The normal at (x', y') has for its equation

$$xx' - yy' = x'^2 - y'^2. \tag{1}$$

By combining (1) and $xy = m^2$, the co-ordinates of the required point are found to be $-\frac{y'^2}{x'}, -\frac{x'^2}{y'}$.

Remark. The normal to $xy = m^2$ at the last point cuts the curve again at $\left(\frac{x'^5}{y'^4}, \frac{y'^5}{x'^4}\right)$.

The equation to the line joining this point and (x', y') is

$$xy'^5 + yx'^5 = x'y'^5 + y'x'^5 = m^2(x'^4 + y'^4).$$

The triangle enclosed by the three lines has an area equal to

$$\frac{(x'^2 + y'^2)(x'^4 - y'^4)(x'^6 + y'^6)}{2x'^5 y'^5}.$$ [*R. H. Graves.*]

179

FIND the area of the triangle formed by the asymptotes to an equilateral hyperbola and the normal to the curve.

SOLUTION.

The equation of the normal referred to the centre and asymptotes is

$$y - y' = \frac{x'}{y'}(x - x').$$

The half product of the values of x when $y = 0$, and y when $x = 0$, gives for the area of the triangle $(x'^2 - y'^2)^2/(2x'y')$. [*H. W. Draughon.*]

189

SHOW how to resolve a given force into three coplanar components acting in given lines not concurrent.

SOLUTION.

Let F be the force, *a point vector*. Since the lines are coplanar and non-current, they form a triangle. Let its vertices be e_1, e_2, e_3, and hence the three lines, e_1e_2, e_2e_3, e_3e_1. Then, if x_1, x_2, x_3 are scalar constants, we have

$$F = x_1e_2e_3 + x_2e_3e_1 + x_3e_1e_2.$$

Multiplying by e_1, $\quad e_1F = x_1e_1e_2e_3$, or $x_1 = \dfrac{e_1F}{e_1e_2e_3}$;

and similarly $\quad x_2 = \dfrac{e_2F}{e_1e_2e_3}$, $\quad x_3 = \dfrac{e_3F}{e_1e_2e_3}$.

Hence $\quad F = \dfrac{1}{e_1e_2e_3}(e_1F \,.\, e_2e_3 + e_2F \,.\, e_3e_1 + e_3F \,.\, e_1e_2)$.

Thus the length of the component on e_1e_2 is the length of e_1e_2 times the ratio of the triangle formed by joining e_3 with the ends of F to the triangle $e_1e_2e_3$; and similarly for other components. [*E. W. Hyde.*]

EXERCISES.

213

FIND the centre of gravity of the area of one quadrant of the tetracuspid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. [*R. H. Graves.*]

214

AN equilateral hyperbola is circumscribed to a triangle. Find the greatest and least values of the transverse axis. [*R. H. Graves.*]

215

Two equal circles, radii r, intersect; find the average area common to both. [*Artemas Martin.*]

216

Two equal spheres, radii r, intersect; find the average volume common to both. [*Artemas Martin.*]

217

A CIRCLE is drawn at random within a given circle. What is the probability that the random circle contains the centre of the given circle? [*Artemas Martin.*]

218

THE lengths of none of three lines exceeds a; find the probability that an acute triangle can be formed with them. [*Artemas Martin.*]

YALE PRIZE PROBLEMS.

SENIOR.

219

IMAGINE a homogeneous elastic string for which the law of elasticity is such that the length of any portion varies as the tension to which it is subjected. A loop of such a string is in equilibrium under the action of a central force, which repels the material of the string inversely as the square of the distance from the centre. What form (other than a circle) may the loop have?

220

A and B are two fixed points 4 units apart. AC and BD are two bars, each 5 units long, pivoted at C and D to the extremities of a third bar 2 units long. What is the deviation from a straight line in the curve described by the middle point of CD while AC and BD cross each other.

221

DISCUSS the remarkable points connected with the spherical triangle, such as the orthocentre, incentre, circumcentre, etc.

222

GIVE the formula for a table of meridional parts for a constant negative curvature ($r = 5$). [If we wish to make a plane map of the surface in which angles shall be preserved, an arbitrarily chosen geodetic line of the surface may be represented in the map by a straight line and on a uniform scale, like the equator in Mercator's chart. Lines equidistant from the geodetic in the surface will be represented by parallel lines in the map, and the formula required is one which will give the distances from the fundamental line at which these parallels are to be drawn.]

223

The points P, Q, R, S are taken on the faces of the tetrahedron $ABCD$, viz. P on the face BCD, etc. The ratios of the triangles PBC, PCD, PDB to the whole face BCD are given, and the corresponding ratios for the other faces. Required the ratio of the tetrahedra $PQRS$ and $ABCD$.

224

A cylinder whose radius is r, lying lengthwise in a rough horizontal cylindrical trough of radius R, receives a slight impact at right angles to its axis; discuss the resulting motion.

225

An ellipse whose axes are $2a$ and $2b$, and whose centre is fixed at A, rolls upon a horizontal plane situated at the distance c below A. Discuss the curve which the ellipse traces upon the plane.

JUNIOR.

226

If the angles of a triangle as computed from slightly erroneous measurements of the sides, are A, B, and C, the errors in the sides being α, β, and γ; prove that the consequent errors in the cotangents of the angles are proportional to

$$\frac{\beta \cos C + \gamma \cos B - \alpha}{\sin A}, \quad \frac{\gamma \cos A + \alpha \cos C - \beta}{\sin B}, \quad \frac{\alpha \cos B + \beta \cos A - \gamma}{\sin C}.$$

227

Write a rule for the extraction of the cube root in duodecimals, and as an example, find to five places the diameter of a sphere which contains one cubic foot.

228

Suppose a vessel in the form of an inverted frustum of a cone to be filled with water, and then tipped until as much water runs out as is possible without uncovering any part of the bottom; required, the volume of the water which remains.

CONTENTS.

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